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1992 J. Phys. A: Math. Gen. 25 401

(http://iopscience.iop.org/0305-4470/25/2/020)

J. Phys. A: Math. Gen. **25** (1992) 401-410. Printed in the UK

The q-deformed boson realization of representations of quantum universal enveloping algebras for q a root of unity: II. The subalgebra chain of $U_q(C_l)$

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Received 15 November 1990, in final form 15 July 1991

Abstract. Applying the method of the q-deformed boson realization for the quantum group to the case where q is a root of unity we establish a set of standard basis for the representation space of the quantum universal enveloping algebra $U_q(C_l)$ associated with a typical subalgebra chain of $U_q(C_l)$. On this basis we systematically obtain irreducible and indecomposable representations of $U_q(C_l)$ and its subalgebras. We discuss $U_q(C_2)$ in detail.

1. Introduction

In this paper, the second part of a series, we will study the quantum universal enveloping algebra (QUEA) [1-8] $(C_l)_q \equiv U_q(C_l)$ when q is a root of unity.

We first review notation and some results for $(A_{l-1})_q$ from the preceding paper [29] which will be used in this paper. As the central part of the q-deformed boson realization, the q-deformed boson algebra is an associative algebra over the complex number field $\mathbb{C}$. It is generated by elements $a_i \equiv a_i^-$, a_i^+, $\hat{N}_i$ and I satisfying the q-deformed commutative relations [13-15, 19-24, 28]

$$\begin{aligned} & a_i a_j^+ = \delta_{ij} q^{\hat{N}_i} + q^{-\delta_{ij}} a_j^+ a_i \\ & [\hat{N}_i, a_j^\pm] = \pm\delta_{ij} a_j^\pm \\ & [a_i^\pm, a_j^\pm] = 0 \end{aligned} \tag{1}$$

where $i, j = 1, 2, \ldots, l$ and $q \in \mathbb{C}$. When $q = 1$, the algebra $\mathscr{B}_q(l)$ becomes the usual boson algebra.

On the q-deformed Fock space $\mathscr{F}_q(l)$,

$$\{|\boldsymbol{m}\rangle \equiv |m_1, m_2, \ldots, m_l\rangle = a_1^{+m_1} a_2^{+m_2} \ldots a_l^{+m_l}|0\rangle| \\ \boldsymbol{m} = (m_1, m_2, \ldots, m_l) \in \mathbb{Z}^{+l}, a_i|0\rangle = \hat{N}_i|0\rangle = 0, i = 1, 2, \ldots, l\}$$

the q-deformed boson realization of the QUEA $(A_{l-1})_q$ is

$$\begin{aligned} & E_i = a_i^+ a_{i+1} \qquad F_i = a_{i+1}^+ a_i \\ & H_i = N_i - N_{i+1} \qquad i = 1, 2, \ldots, l-1 \end{aligned} \tag{2}$$

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where

$$\mathbb{Z}^{+l} = \{(m_1, m_2, \ldots, m_l) = \boldsymbol{m} | m_i \in \mathbb{Z}^+ = \{0, 1, 2, \ldots\}, i = 1, 2, \ldots, l\}$$

is the l-dimensional lattice set. Because

$$\left[E_i, \sum_{i=1}^{l} N_i\right] = \left[F_i, \sum_{i=1}^{l} N_i\right] = \left[H_i, \sum_{i=1}^{l} N_i\right] = 0$$

there explicitly exists a finite-dimensional $(\mathrm{A}_{l-1})_q$-invariant subspace in $\mathscr{F}_q(l)$. This is the starting point of the discussion in the preceding paper [29]. A q-deformed boson realization $\{X_i\}$ of a QUEA such that $[X_i, \Sigma_{i=1}^{l} N_i] = 0$ is called a homogeneous q-deformed boson realization; conversely, if $[X_i, \Sigma_{i=1}^{l} N_i] \neq 0$, then the realization $\{X_i\}$ is called an inhomogeneous q-deformed boson realization.

However, a homogeneous q-deformed boson realization of QUEA $(\mathrm{C}_l)_q$ cannot be found. We need some skill to obtain finite-dimensional representations of $(\mathrm{C}_l)_q$ through its inhomogeneous q-deformed boson realization when $q^p = 1$ and p is an integer $\geqslant 3$. In this paper, associated with a subalgebra chain of $(\mathrm{C}_l)_q$, a set of standard basis for the representation of $(\mathrm{C}_l)_q$ is well defined so that the reduction of representation is automatically realized when a representation of an algebra is restricted on its subalgebras in this subalgebra chain. In particular, the positive integers $\lambda_l, \lambda_{l-1}, \ldots$ and λ_2 in the index $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of the basis for a representation of $(\mathrm{C}_l)_q$ label the irreducible representations of $(\mathrm{A}_{l-1}), (\mathrm{A}_{l-2}), \ldots$ and (A_1), respectively, when $q^p \neq 1$ $(p = 1, 2, \ldots)$.

2. The q-deformed boson realization of the $(\mathrm{C}_l)_q$-subalgebra chain

The QUEA $(\mathrm{C}_l)_q$ is an associative algebra (over $\mathbb{C}$) generated by E_i, F_i and H_i $(i = 1, 2, \ldots, l)$ satisfying the q-commutation relations

$$\begin{aligned} &[H_i, E_j] = \alpha_{ij} E_j && [H_i, F_j] = -\alpha_{ij} F_j \\ &[H_i, H_j] = 0 && [E_i F_j] = \delta_{ij} [H_i]_{q_i} \end{aligned} \tag{3}$$

and the Serre relations

$$\begin{aligned} &G_j^2 G_{j+1} - (q + q^{-1}) G_j G_{j+1} G_j + G_{j+1} G_j^2 = 0 && 1 \leqslant j \leqslant l-2 \\ &G_j^2 G_{j-1} - (q + q^{-1}) G_j G_{j-1} G_j + G_{j-1} G_j^2 = 0 && 1 \leqslant j \leqslant l-1 \\ &\sum_{m=0}^{3} (-1)^m [3]_{q^2} ([m]_{q^2}! [3-m]_{q^2}!)^{-1} G_{l-1}^{3-m} G_l G_{l-1}^m = 0 \\ &G = E, F \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\alpha_{ij} = 2\delta_{ij} - \delta_{ij+1} - \delta_{ij-1} \qquad \alpha_{il} = -2\delta_{i,l-1} \qquad \alpha_{li} = -\delta_{il-1} \\ &\alpha_{l,l} = l \qquad i, j = 1, 2, \ldots, l-1 \qquad [f]_t = (t^f - t^{-f})(t - t^{-1})^{-1} \\ &q_i = q \in \mathbb{C} (q \leqslant i \leqslant l-1) \qquad q_l = q^2 \\ &[m]_t! = [m]_t [m-1]_t \ldots [2]_t [1]_t \qquad [f]_{t=q} \equiv [f]. \end{aligned}$$

In fact, E_i, F_i and H $(i = 1, 2, \ldots, k)$ are closed under the operation of the q-commutators and the relations of $(\mathrm{A}_k)_q$ [29] are satisfied by them for $k = 1, 2, \ldots, l-1$. They generate a subalgebra $(\mathrm{A}_k)_q$.

The q-deformed boson realization of a QUEA $U_q(L):\{x_i\}$ is the image $B(U_q(L)):\{x_i \equiv B(x_i)\}$ of an isomorphic mapping $B: U_q(L) \to \mathscr{B}_q(l)$. By taking the q-deformed boson realization of $(A_{l-1})_q$ [14, 29] into account, the q-deformed boson realization of $(C_l)_q$ is written as [28]

$$\begin{aligned} &E_i = a_i^+ a_{i+1} \qquad F_i = a_{i+1}^+ a_i \qquad H_i = \hat{N}_i - \hat{N}_{i+1} \qquad i = 1, 2, \ldots, l-1 \\ &E_l = \frac{1}{[2]} a_l^{+2} \qquad F_l = -\frac{1}{[2]} a_l^2 \qquad H_l = N_l + \tfrac{1}{2}. \end{aligned} \tag{5}$$

We can check that relations (3) and (4) hold for the above realization due to the basic relations (1). We need to point out that realization (5) is only an alternative to the result of [19]. However, we make a simple but important observation that the realizations of E_i, F_i and H_i also satisfy the q-commutation relations and the Serre relations of $(A_k)_q$ for $1, 2, \ldots, k$, where the fixed k takes $1, 2, \ldots, l-1$. Thus, realization (5) actually defines a realization of subalgebra chain I:

$$(C_l)_q \supset (A_{l-1})_q \supset (A_{l-2}) \supset \ldots \supset (A_2)_q \supset (A_1)_q.$$

Because E_j, F_j and H_j $(j = k, k+1, \ldots, l-1)$ generate a subalgebra (C_{l-k+1}) of $(C_l)_q$, (3) also gives a realization of subalgebra chain II:

$$(C_l)_q \supset (C_{l-1})_q \supset (C_{l-2})_q \supset \ldots \supset (C_3)_q \supset (C_2)_q.$$

The basic subalgebra chains I and II can derive other subalgebra chains, e.g.

$$\begin{array}{llll} & \supset\ (A_{l-1}(1))_q \supset (A_{l-2}(1))_q \supset \ldots \supset (A_1(1))_q & & \\ (C_l)_q \supset & (C_{l-1})_q \supset (C_{l-2})_q \supset \ldots \supset (C_2)_q & & \\ & \cup \qquad\qquad \cup & & \\ & \qquad\qquad (A_{l-3}(3))_q \supset \ldots \supset (A_1(3))_q & & \\ & (A_{l-2}(2))_q \supset (A_{l-3}(2))_q \supset \ldots \supset (A_1(2))_q & & \end{array}$$

where t in $(A_k(t))_q$ denotes the different embedding of $(A_k)_q$ in $(C_l)_q$, e.g.

$$\begin{aligned} &(A_1(1))_q = \{H_1, E_1, F_1\} \\ &(A_1(2))_q = \{H_2, E_2, F_2\} \\ &\ldots \end{aligned}$$

Since the realization of the subalgebra $(A_k)_q$ in chain I is homogeneous, we will build up a standard basis associated with this chain so that finite-dimensional representations are easily obtained in this paper.

3. Representations of subalgebra chain I of $(C_l)_q$

Using (1) and (3), on the q-deformed Fock space $\mathscr{F}_q(l)$, we obtain a representation Γ_l of $(C_l)_q$:

$$\begin{aligned} &E_i|\boldsymbol{m}\rangle = [m_{i+1}]|\boldsymbol{m} + \boldsymbol{e}_i(l) - \boldsymbol{e}_{i+1}(l)\rangle \\ &F_i|\boldsymbol{m}\rangle = [m_i]|\boldsymbol{m} - \boldsymbol{e}_i(l) + \boldsymbol{e}_{i+1}(l)\rangle \\ &H_i|\boldsymbol{m}\rangle = (m_i - m_{i+1})|\boldsymbol{m}\rangle \qquad i = 1, 2, \ldots, l-1 \\ &E_l|\boldsymbol{m}\rangle = [2]^{-1}|\boldsymbol{m} + 2\boldsymbol{e}_l(l)\rangle \\ &F_l|\boldsymbol{m}\rangle = -[2]^{-1}[m_{l-1}][m_l]|\boldsymbol{m} - 2\boldsymbol{e}_l(l)\rangle \\ &H_l|\boldsymbol{m}\rangle = (m_l + \tfrac{1}{2})|\boldsymbol{m}\rangle \end{aligned} \tag{6}$$

where

$$e_l(1)=(1,0,\ldots,0)$$
$$e_l(2)=(0,1,\ldots,0),\ldots,e_l(l)=(0,0,\ldots,1).$$

It follows from (6) that $(-1)^{\sum_{i=1}^{l} m_i}$ is invariant under the action of Γ_l and so this representation Γ_l is reduced a direct sum of two representations Γ_l^+ and Γ_l^-, i.e.

$$\Gamma_l=\Gamma_l^+\oplus\Gamma_l^-.$$

Correspondingly, the reduction of the space $\mathscr{F}_q(l)$ is

$$\mathscr{F}_q(l)=\mathscr{F}_q^+(l)\oplus\mathscr{F}_q^-(l)$$
$$\mathscr{F}_q^\pm(l):\{|\boldsymbol{m}\rangle|(-1)^{\sum_{i=1}^{l} m_i}=\pm 1\}.$$

The discussion on $\mathscr{F}_q^-(l)$ parallels that on $\mathscr{F}_l^+(l)$ and so the following discussion only focuses on $\mathscr{F}_q^+(l)$. Define the standard basis

$$f(\boldsymbol{\lambda}|J)=|\lambda_1,\lambda_2-\lambda_1,\lambda_3-\lambda_2,\ldots,\lambda_{l-1}-\lambda_{l-2},2J-\lambda_{l-1}\rangle$$

for $\mathscr{F}_q^+(l)$ where $J=1,2,\ldots,\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_{l-1})\in\mathbb{Z}^{+l-1}$ and $\lambda_{k-1}=0,1,2,\ldots$ for given λ_k ($k=2,3,4,\ldots,;\lambda_l=2J$). Let $\lambda_0=0$. Then, on the above new basis, representation (6) is rewritten as

$$E_if(\boldsymbol{\lambda}|J)=[\lambda_{i+1}-\lambda_i]f(\boldsymbol{\lambda}+e_i(l-1)|J) \tag{7a}$$
$$F_if(\boldsymbol{\lambda}|J)=[\lambda_i-\lambda_{i-1}]f(\boldsymbol{\lambda}-e_i(l-1)|J) \tag{7b}$$
$$H_if(\boldsymbol{\lambda}|J)=(2\lambda_i-\lambda_{i-1}-\lambda_{i+1})f(\boldsymbol{\lambda}|J) \qquad i=1,2,\ldots,l-1 \tag{7c}$$
$$E_lf|\boldsymbol{\lambda}|J)=[2]^{-1}f(\boldsymbol{\lambda}|J+1) \tag{7d}$$
$$F_lf(\boldsymbol{\lambda}|J)=-[2][2J-\lambda_{l-1}][2J-\lambda_{l-1}-1]f(\boldsymbol{\lambda}|J-1) \tag{7e}$$
$$H_lf(\boldsymbol{\lambda}|J)=(2J-\lambda_{l-1}+\tfrac{1}{2})f(\boldsymbol{\lambda}|J). \tag{7f}$$

To show the characters of the standard basis $f(\boldsymbol{\lambda}|J)$ we tentatively consider the case where q is not a root of unity. In this case, the representation $\Gamma_l^+:(\mathrm{C}_l)_q\to\mathrm{End}(\mathscr{F}_q^+(l))$ is irreducible. Because J is invariant under the action of $(\mathrm{A}_{l-1})_q$ through the representation Γ_l^+, it labels an irreducible representation $\Gamma_l^{[2J]}$ of $(\mathrm{A}_{l-1})_q$ on the invariant subspace

$$V_l^{[2J]}:\{f(\boldsymbol{\lambda}|J)\in\mathscr{F}_q^+(l)\}\simeq\left\{|m_1,m_2,\ldots,m_l\rangle|\sum_{k=1}^{l}m_k=2J\right\}$$

for a fixed J. When the irreducible representation $\Gamma_l^{[2J]}$ of $(\mathrm{C}_l)_q$ is restricted on its subalgebra $(\mathrm{A}_{l-1})_q$ there is an automatic reduction:

$$\Gamma_l^+|(\mathrm{A}_{l-1})_q=\sum_{J=0}^{\infty}\oplus\,\Gamma_l^{[2J]}$$

$$\mathscr{F}_q^+(l)|=\sum_{J=0}^{\infty}\oplus\,V_l^{[2J]}.$$

Similarly, λ_k labels an irreducible representation $\Gamma_k^{[\lambda_k]}$ of $(\mathrm{A}_{k-1})_q$ on the invariant subspace

$$V_k^{[\lambda k]}:\left\{f(\boldsymbol{\lambda}|J)\equiv f_{\lambda k}(\lambda_1,\ldots,\lambda_{k-1})\simeq|\boldsymbol{m}\rangle|\boldsymbol{m}\in\mathbb{Z}^{k-1},\sum_{i=1}^{k-1}m_i=\lambda_k\right\}$$

for fixed λ_k, $\lambda_{k+1}, \ldots$ and λ_l. Correspondingly, we have

$$\Gamma_{k+1}^{[\lambda_{k+1}]}|(A_{k-1})_q = \sum_{\lambda_k=0}^{\lambda_{k+1}}\oplus\ \Gamma_k^{[\lambda_k]}$$

$$V_{k+1}^{[\lambda_{k+1}]} = \sum_{\lambda_k=0}^{\lambda_{k+1}}\oplus\ V_k^{[\lambda_k]}.$$

4. A theorem on indecomposable representations of $(C_l)_q$

Now, we return to the discussion of the case where q is a root of unity. In this case the above reduction and decomposition of a representation will still be preserved, but some invariant subspaces for $q^p \neq 1$ are no longer invariant while the irreducible representions carried on them are no longer irreducible. The foregoing results from $[\alpha p] = 0$ ($\alpha \in \mathbb{Z}$) when $q^p = 1$. Correspondingly, there exist such extreme weight vectors $f(\underline{\lambda}|\underline{J})(\underline{\lambda}_{i+1} - \underline{\lambda}_i = \alpha_i p, \alpha_i \in \mathbb{Z})$ that $E_i f(\underline{\lambda}|\underline{J}) = F_{i+1} f(\underline{\lambda}|\underline{J}) = 0$.

Theorem 1. In $\mathscr{F}_q^+(l)$ there exists a $(C_l)_q$-invariant subspace $S_l(j, \alpha)$ spanned by the weight vectors

$$\{f(\boldsymbol{\lambda}|J) | \lambda_{j+1} - \lambda_j \geqslant \alpha p, J \in \mathbb{Z}^+\} \qquad \alpha \in \mathbb{Z}^+.$$

When $l \geqslant 2$, there is not an invariant complementary subspace for $S_l(j, \alpha)$; that is to say, the representation Γ_l^+ of $(C_l)_1$ is indecomposable (reducible, but not completely reducible).

Proof. For $k \in \mathbb{Z}^+$ we define a subspace $W(j, k): \{f(\boldsymbol{\lambda}|J) \in \mathscr{F}_q^+(l) | \lambda_{j+1} - \lambda_j = k\}$ of $\mathscr{F}_q^+(l)$. Then,

$$S_l(j, \alpha) = \sum_{k=\alpha p}^{\infty}\oplus\ W(j, k).$$

It follows from $(7a)$ and $(7b)$ that

$$\begin{aligned} E_{j+1} f(\boldsymbol{\lambda}|J) &= [\lambda_{j+2} - \lambda_{j+1}] f(\boldsymbol{\lambda} + e_{j+1}(l-1)|J) \\ F_{j+1} f(\boldsymbol{\lambda}|J) &= [\lambda_{j+1} - \lambda_j] f(\boldsymbol{\lambda} - e_{j+1}(l-1)|J). \end{aligned} \tag{8}$$

From (8), $(7a)$ and $(7b)$ we observe that

$$\begin{aligned} &E_{j+1} f(\boldsymbol{\lambda}|J) \in W(j, k+1) \subset S_l(j, \alpha) \\ &F_j f(\boldsymbol{\lambda}|J) \in W(j, k+1) \subset S_l(j, \alpha) \qquad (j \neq l-1) \\ &E_i f(\boldsymbol{\lambda}|J) \in W(j, k) \subset S_l(j, \alpha) \\ &F_i f(\boldsymbol{\lambda}|J) \in W(j, k) \subset S_l(j, \alpha) \qquad i \neq j, j+1 \\ &E_l W(l-1, k) \subset W(l-1, k) \subset S_l(l-1, \alpha) \end{aligned}$$

for $f(\boldsymbol{\lambda}|J) \in W(j, k) \subset S_l(j, \alpha)$. Due to $[\lambda_{j+1} - \lambda_j] = [\alpha p] = 0$,

$$E_j f(\boldsymbol{\lambda}|J) = F_{j+1} f(\boldsymbol{\lambda}|J) = 0$$

for $f(\boldsymbol{\lambda}|J) \in W(j, \alpha p)$. For $f(\boldsymbol{\lambda}|J) \in W(j, k)$ and $k = \alpha p + 1, \alpha p + 2 \ldots$,

$$\begin{aligned} &E_j f(\boldsymbol{\lambda}|J) \in W(j, k-1) \subset S_l(j, \alpha) \\ &F_{j+1} f(\boldsymbol{\lambda}|J) \in W(j, k-1) \subset S_l(j, \alpha) \qquad j \neq l-1 \\ &F_l W(l-1, \alpha p + 1) = \{0\} \\ &F_l W(l-1, k) \subset W(l-1, k-2) \subset S_l(l-1, \alpha) \qquad k \geqslant \alpha p + 2. \end{aligned}$$

The above analysis shows that for any $f(\lambda|J) \in S_l(j, \alpha)$, $E_i f(\boldsymbol{\lambda}/J)$, $F_i f(\boldsymbol{\lambda}|J)$, $H_i f(\boldsymbol{\lambda}|J) \in S_l(j, \sigma)$; that is to say, $S_l(j, \alpha)$ is invariant.

Now, we prove that there is not an invariant complementary space for $S_l\ (j, \alpha)$. Assume that there is an invariant subspace $\bar{S}_l\ (j, \alpha)$ so that $\mathscr{F}_q^+(l) = S_l(j, \alpha) \oplus \bar{S}_l(j, \alpha)$. There must exist a vector

$$v = \sum_{\lambda_{j+1}-\lambda_j \geqslant \alpha p} c_{\boldsymbol{\lambda}} f(\boldsymbol{\lambda}|J) + \sum_{\lambda_{j+1}-\lambda_j < \alpha p} b_{\boldsymbol{\lambda}} f(\boldsymbol{\lambda}|J) = v_1 + v_2$$

which has a $c_{\boldsymbol{\lambda}} \neq 0$ and a $b_{\boldsymbol{\lambda}} \neq 0$ at least. Let $b_{\boldsymbol{\lambda}}$ correspond to such a vector $f(\boldsymbol{\lambda}|J)$ that $\underline{k} = \underline{\lambda}_{j+1} - \underline{\lambda}_j$ is a minimum. Then,

$$(E_{j+1})^{\alpha p - \underline{k}} v_2(\neq 0) \in S_l(j, \alpha) \qquad (E_{j+1})^{\alpha p - \underline{k}} v(\neq 0) \in S_l(j, \alpha).$$

Due to this presumption

$$(E_{j+1})^{\alpha p - \underline{k}} v \in \bar{S}_l(j, \alpha).$$

Thus,

$$S_l(j, \alpha) \bar{\cap} S_l(j, \alpha) \neq \{0\}.$$

This conclusion and the presumption are contradictory. □

A corollary immediately follows from theorem 1.

Corollary. When $q^p = 1$, there exists an (A_{l-1})-invariant subspace $S_l^{[J]}(j, \alpha)$ spanned by the vectors

$$\{f(\boldsymbol{\lambda}|J) | \lambda_{j+1} - \lambda_j \geqslant \alpha p\}$$

in $V_l^{[2J]}$ for fixed J. When $l > 2$, there is not an invariant complementary space for $S_l^{[J]}(j, \alpha)$; the representation $\Gamma_l^{[2J]}$ is indecomposable.

In fact, this corollary is theorem 1 of the preceding paper [29] and some discussions in the preceding paper can be regarded as special cases of this present paper. Therefore, in the following discussions we are no longer concerned with the subalgebra $(\mathrm{A}_k)_q$ of $(\mathrm{C}_l)_q$.

5. Finite-dimensional representations of $(\mathrm{C}_2)_q$

According to the above general analysis, we study the representation Γ_2^+ of $(\mathrm{C}_2)_q$:

$$\begin{aligned}
E_1 f(n|J) &= [2J - n] f(n+1|J) \\
F_1 f(n|J) &= [n] f(n|J) \\
H_1 f(n|J) &= 2(n - J) f(n|J) \\
E_2 f(n|J) &= [2]^{-1} f(n|J+1) \\
F_2 f(n|J) &= -[2]^{-1}[2J - n][2J - n - 1] f(n|J-1) \\
H_2 f(n|J) &= (2J - n + \tfrac{1}{2}) f(n|J)
\end{aligned} \tag{9}$$

where $f(n|J) = a_1^{+n} a_2^{2J-n}|0\rangle$ and for a given J, $n = 0, 1, 2, \ldots, 2J$. This representation is illustrated in figure 1 where a lattice (J, n) denotes a weight vector $f(n|J)$, the upward, downward, rightward and leftward arrows denote the actions of E_1, F_1, E_2 and F_2, respectively.

The character lines l_1: $n=\beta p$ and l_2: $2J-n=\alpha p$ cut two invariant subspaces

$$S_2(1,\beta)(\beta\in\mathbb{Z}^+):\{f(n|J)\in\mathscr{F}_q^+(2)|n\geqslant\beta p\}$$

$$S_2(2,\alpha)(\alpha\in\mathbb{Z}^+):\{f(n|J)\in\mathscr{F}_q^+(2)|2J-n\geqslant\alpha p\}$$

out of $\mathscr{F}_q^+(2)$ (see figures 2 and 3). The representation Γ_2^+ subduces new representations on the two spaces, which are still infinite dimensional. However, the sum $S_{12}(\alpha,\beta)=S_2(1,\beta)+S_2(2,\alpha)$ is invariant under Γ_2^+. So there is a quotient space

$$Q_{12}(\alpha,\beta)=\mathscr{F}_q^+(2)/S_{12}(\alpha,\beta):\{\bar f(n|J)=f(n|J)\bmod S_{12}(\alpha,\beta)|$$
$$0\leqslant 2J-n\leqslant\alpha p-1, 0\leqslant n\leqslant\beta p-1\}$$

illustrated as the unshaded domain in figure 4. Then, we have the following theorem.

Theorem 2. On the quotient space $Q_{12}(\alpha,\beta)$ the representation Γ_2^+ induces a finite-dimensional representation of $(\mathrm{C}_2)_q$ with dimension

$$\dim Q_{12}(\alpha,\beta)=\tfrac{1}{2}(\alpha\beta p^2+\sigma(\alpha p)\sigma(\beta p)) \tag{10}$$

where $\sigma(x)=\frac{1}{2}(1-(-1)^x)$.

Proof. For a given βp, n takes βp values $0,1,2,\ldots,p-1$ for each n, because J is an integer,

$$n/2\leqslant J\leqslant(\alpha p+n-1)/2$$

and J takes $\frac{1}{2}(\alpha p+\sigma(\alpha p)$ values for even J and $\frac{1}{2}(\alpha p-\sigma(\alpha p))$ values for odd J. Since there are $\frac{1}{2}(\beta p+\sigma(\beta p))$ even integers and $\frac{1}{2}(\beta p-\sigma(\beta p))$ odd integers in the chain

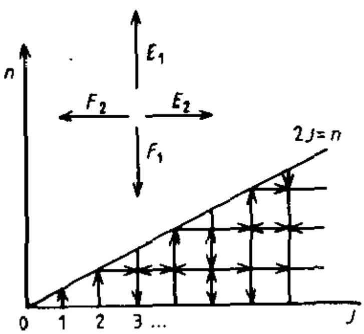

Figure 1. The representation space.

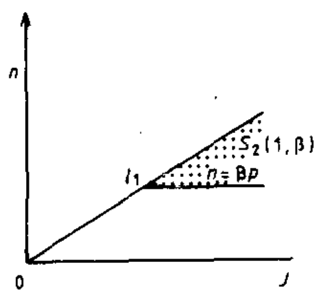

Figure 2. The invariant subspace $S_2(1,\beta)$.

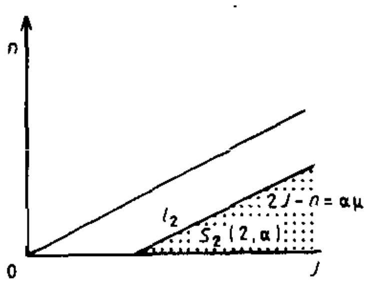

Figure 3. The invariant subspace $S_2(2,\alpha)$.

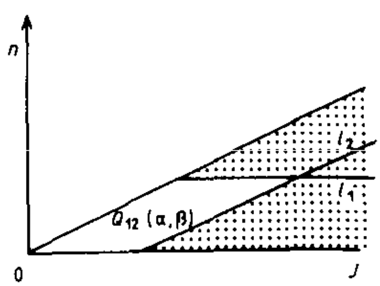

Figure 4. The quotient space $Q_{12}(\alpha,\beta)$.

$0, 1, 2, 3, \ldots, p-1$, the pair (J, n) takes

$$\tfrac{1}{2}(\beta p+\sigma(\beta p))\tfrac{1}{2}(\alpha p+\sigma(\alpha p))+\tfrac{1}{2}(\beta p-\sigma(\beta p))\tfrac{1}{2}(\alpha p-\sigma(\alpha p))$$
$$=\tfrac{1}{2}(\alpha\beta p^2-\sigma(\alpha p)\sigma(\beta p))\equiv \dim Q_{12}(\alpha\beta)$$

values; i.e. the dimension of the quotient space is $\dim Q_{12}(\alpha\beta)$. □

Naturally, on the quotient space $Q_{12}(\alpha\beta)$, Γ_2^+ induces a finite dimensional representation $\bar{\Gamma}_2^+$: $(\mathrm{C})_q \to \mathrm{End}(Q_{12}(\alpha, \beta))$:

$$g\bar{f}(n|J)\equiv\bar{\Gamma}_2^+(g)\bar{f}(n|J)=\overline{\Gamma_2^+(g)f(n|J)}\equiv\overline{gf(n|J)}.$$

This induced representation is determined in an explicit form by (9) and such extreme vectors $f(\beta p-1|J)$ and $f(2J+1-\alpha p|J)$ that

$$E_1\bar{f}(\beta p-1|J)=E_2\bar{f}(2J+1-\alpha p|J)=0.$$

As an example, when $p=3$ and $\alpha=\beta=1$ we obtain a five-dimensional irreducible representation

$$\begin{aligned}
&E_1=E_{43}+[2]E_{32} \qquad && E_2=[2]^{-1}(E_{21}+E_{54})\\
&F_1=E_{23}+[2]E_{34} && F_2=-(E_{12}+E_{45})\\
&H_1=2(E_{44}-E_{22}) && \\
&H_2=\tfrac{1}{2}(E_{11}+5E_{22}+3E_3+E_{44}+5E_{55}) &&
\end{aligned}\tag{11}$$

where E_{ij} is a unit of a matrix, i.e. $(E_{ij})_{kl}=\delta_{ik}\delta_{jl}$.

Finally, we consider an invariant subspace chain

$$\mathcal{F}_q^+(2)\supset S_{12}(1,1)\supset S_{12}(1,2)\supset\ldots\supset S_{12}(1,i)\supset\ldots.$$

On each quotient space $\bar{Q}_i=S_{12}(1,i)/S_{12}(1,i+1)$,

$$\{\underline{f}(n|J)=f(n|J) \bmod S_{12}(1,i+1)|0\leqslant 2J-n\leqslant p-1,\ ip\leqslant n\leqslant(i+1)p\}$$

Γ_2^+ induces a quotient representation $\Gamma_2^{[i]+}$. Then, the representation Γ_2^+ is reduced to a semi-direct sum of $\Gamma_2^{[i]+}$, i.e.

$$\Gamma_2^+=\Gamma_2^{[1]+}\bar{\oplus}\Gamma_2^{[2]+}\bar{\oplus}\ldots\bar{\oplus}\Gamma_2^{[i]}\oplus\ldots.$$

6. Discussion

Up to now we have explicitly constructed a class of representations of the quantum algebra $(\mathrm{C}_l)_q$, which contains both irreducible and indecomposable representations. The status of the results in this paper and the relevant open questions are discussed as follows.

(i) Although, from the generally mathematical point of view, the representation theories of quantum groups (quantum algebras) have been built for both generic [9, 10] and the non-generic cases [11, 12, 30-32], it is still necessary to explicitly give the representations in matrix form for the needs in physics [25–27]. In this paper we first obtained a class of symmetric tensor representations in explicit form. Our results not only contain irreducible representations (e.g. the $\frac{1}{2}(p^2-(\sigma(p))^2)$]-dimensional representations for $(\mathrm{C}_2)_q$), as explicit examples of the general theory, but also contain indecomposable representations (e.g. the representations on the spaces $Q_{1,2}(\alpha,\beta)$-$(\alpha,\beta\geqslant 2)$ for $(\mathrm{C}_2)_q$), which is completely new. In particular, the introduction of the analogue of boson operators not only simplifies the process of constructing the representations, but is also formally familiar to physicists.

Considering our recent studies in which new solutions of the Yang–Baxter equation were constructed from a new family of explicit representations for $sl_q(2)$ [33], we hope that the representations of $(C_l)_q$ obtained in this paper with the explicit forms can also be used for the construction of other new solutions for the Yang–Baxter equation.

It is necessary to point out that we have not obtained all of the irreducible representations of $(C_l)_q$. As for $(A_l)_q$, we only obtained a class of symmetrical representations because only l boson degrees were used for the l-rank quantum algebra $(C_l)_q$. How to generalize the q-deformed boson realization method to get all the irreducible representations of $(C_l)_q$ is still an open question. Maybe the q-analogue of the Borel–Weil construction for Lie algebras [34] is needed for this question.

(ii) It worth noticing that, when $q \to 1$, realization (3) becomes an operator representation of the Lie algebra C_l and contains a subalgebra su(1, 1) generated by E_l, F_l and H_l. Because this subalgebra is non-compact, realization (3) naturally results in infinite-dimensional irreducible representations for both the Lie algebra C_l as the limit for $q \to 1$ and the quantum algebra $(C_l)_q$ with generic q $(q^p \neq 1)$. Such a realization associated with non-compactness is called a non-compact boson realization of quantum algebra and usually given an infinite-dimensional non-unitary irreducible representation for the generic case.

For the non-generic case the fact that finite-dimensional representations of $(C_l)_q$ can be obtained from its infinite-dimensional representation by taking $q^p = 1$ shows a completely 'quantum' picture without the 'classical' limit. In fact, as $q \to 1$, the infinite-dimensional representation becomes a representation of the corresponding Lie algebra, but the finite-dimensional representations do not make sense. Such a finite-dimensional representation of the quantum algebra $(L)_q$ cannot regarded as a simple q-deformation of a representation of the corresponding Lie algebra [16–18].

(iii) The results of this paper highlight a q-deformation scheme by which such a completely 'quantum' finite-dimensional representation for the quantum algebra can be constructed. (*a*) For a given Lie algebra L, we try to find a non-compact boson realization of L and then use it to construct an infinite-dimensional representation, which is usually irreducible and non-unitary. (*b*) We perform a q-deformation of this realization so that it becomes a q-boson realization of the quantum algebra $(L)_q$ associated with L. Correspondingly, the representation of L is deformed into an infinite-dimensional representation of $(L)_q$, which is irreducible when q is not a root of unity. (*c*) Taking the non-generic condition $[\alpha p] = 0$ $(\alpha \in \mathbb{Z})$ caused by $q^p = 1$ into account, we can find a finite-dimensional invariant subspace, on which we obtain a finite-dimensional representation of the quantum algebra $(L)_q$. As for $(C_l)_q$, the $(A_l)_q$ case can also be successfully handled using this scheme [25].

Acknowledgments

One of the authors, C P Sun, thanks Dr X F Liu for useful discussions. This work is supported in part by the National Nature Science of Foundation of China.

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